and that $\phi \to 0$ *when* $w \to 0$ *along either of two radii vectores interior to* Δ. *Then* $\phi \to 0$ *uniformly when* $w \to 0$ *in any manner within any region* Δ' *interior to* Δ.

6.3. Similar arguments, based upon Lemma 5, lead to the conclusion that *in Montel's theorem of* 6.1, *the hypothesis that* $\phi(w)$ *is bounded may be replaced by the hypothesis that the integral* (1.15) *is convergent.* A direct proof of this theorem is not without interest, but the actual result is included in the more general theorems referred to in 6.1.*

On the Free Transverse Vibrations of a Uniform Circular Disc Clamped at its Centre; and on the Effects of Rotation.

By R. V. SOUTHWELL, M.A.

(Communicated by Prof. H. Lamb, F.R.S. Received September 23, 1921.)

INTRODUCTION AND SUMMARY.

§ 1. A thin circular disc, entirely unconstrained, can vibrate transversely in modes which may be characterized by any number of nodal concentric circles, crossed by uniformly spaced nodal diameters. The natural frequencies, for a disc of uniform thickness, have been calculated by Kirchhoff,† and a recent investigation by Prof. Lamb and the present author‡ has shown the amount by which they are increased when the disc rotates at any given speed about its axis.

This extension of Kirchhoff's analysis was suggested by published accounts of failures in turbine discs, the circumstances of which indicated that they had their origin in forced vibrations, intensified by resonance.§ It was realised that the results would be only indirectly applicable to design, mainly because in practice discs are constructed with curved profiles (the thickness increasing from the rim to the boss), but also because the calculated frequencies, in some instances, will be seriously affected by the "clamping" which is introduced when the disc is attached at its centre to a shaft and rotor of considerable inertia.

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* For $\phi(w)$ covers a finite area when w covers Δ, and there are therefore infinitely many values which $\phi(w)$ does not take.

† "Ueber die Schwingungen einer elastischen Scheibe," 'Crelle's Journal,' vol. 40 (1850), and 'Pogg. Ann.,' vol. 81 (1850). The essentials of the analysis are reproduced in Lord Rayleigh's 'Theory of Sound,' vol. 1, §§ 218, 219.

‡ 'Roy. Soc. Proc.,' A, vol. 99, pp. 272–280 (1921). Throughout the present investigation this paper will be quoted by the abbreviated title "Spinning Discs," and its equations by their number with an A affixed: thus equation (34A) below is equation (34) of the "Spinning Discs" paper.

§ *Cf.* K. Baumann, "Some Recent Developments in Large Steam Turbine Practice," Journ. Inst. Elect. Eng.,' vol. 59, pp. 619, 620 (1921).

Varying thickness, and the allied complication of loads imposed at the rim by the inertia of the blading, introduce difficulties of such magnitude that it will probably not be possible to calculate the effects of these factors with any great accuracy. But the effect of central clamping may be studied in a general fashion, if we confine our attention, as before, to a disc of uniform thickness, and calculate, by a further extension of Kirchhoff's analysis, the increase in frequency which results when a constraint of this nature is introduced. The present paper contains the results of an investigation on these lines. Its scheme is generally similar to that of the joint paper to which reference has been made; that is to say, exact calculations are given for the two limiting cases obtained by neglecting in turn the centrifugal and the flexural stress-systems, and these are followed by an approximate treatment of the problem in its general form.

The assumption made in regard to "central clamping" requires some explanation. It is evident that in practice any alteration in the dimensions of the boss will modify not only the boundary conditions imposed upon the transverse displacement, but also the centrifugal stress-system, which in part determines the differential equation governing this displacement at all points in the disc. Both of these effects will influence the natural frequencies of vibration, the first in a way which we may regard as direct, and the second indirectly. Now the centrifugal stress-system, in practice, will depend upon many factors (*e.g.*, the diameter of the boss, the diameter of the shaft, the stresses imposed by the process of "shrinking on," the weight of the blading, etc.), and it is not possible to make any one assumption in regard to its distribution which will satisfactorily represent all cases. For the purpose of the present paper, which is intended to throw light on the general principles involved, it has therefore been thought best to concentrate attention upon the *direct* consequences of clamping, and to assume that the clamps prevent transverse displacement without altering the centrifugal system, so that this system is such as obtains in a disc which is continuous up to the centre.

A constraint of this nature is easily visualized (*e.g.*, clamps which are brought into contact with the rotating disc *after it has taken up the strains appropriate to its speed of rotation*), and it is easy to reproduce in experiments. The approximate methods given in the concluding section of this paper enable us to deal with any other system of centrifugal stresses which may be specified, and in our exact calculations (§ 3) for the spinning membrane (or disc of negligible flexural rigidity), by retaining unchanged the differential equation used previously, we are led to the interesting conclusion that in this limiting case clamping has no direct effect upon the frequencies, although it alters the modes by introducing terms which would become infinite at the centre; it follows that the only way in which clamps can modify the frequencies is indirectly, by changing the centrifugal stress-system.

The other limiting case, of the non-rotating disc, is discussed in § 2. Figs. 3–6 show how the frequency varies with the radius of clamping in the gravest modes having 0, 1, 2 or 3 nodal diameters, and some of the higher frequencies have been calculated for the special case in which the radius of the clamping circle is infinitesimal. It was to be expected that a constraint of this particular type would involve no alteration in frequency for modes having two or more nodal diameters, since in such modes both the displacement and the slope vanish at the centre even when the disc is completely free. It is also found, somewhat unexpectedly, to be without effect in modes having one nodal diameter,* although the frequency in the gravest mode rises very rapidly with an increase in the radius of the clamping circle; but for the "symmetrical" modes, in which there are no nodal diameters, an entirely new series of frequencies is obtained when the centre of the disc is clamped.

In the general case, when the flexural and centrifugal systems are both operative, the conclusion is reached that clamping confined to a small circle has a negligible influence upon the gravest frequencies in modes which have two or more nodal diameters, and that sufficiently exact estimates of these frequencies can be obtained, as explained in the "Spinning Discs" paper, from the relation

$$p^2 = p_1^2 + p_2^2, \tag{34A}$$

where p_1 and p_2 are the gravest values of p found by neglecting (1) the flexural rigidity and (2) the rotation. This formula gives an estimate which is of the nature of a *lower limit* to the true value. It also yields reasonably accurate results as applied to modes having one nodal diameter, if p_2 is given the value appropriate to the size of clamping circle considered. But in the symmetrical modes the gravest frequency found by neglecting the flexural rigidity is zero, the effect of the central constraint being *nil*; hence, the formula (34A), as applied to this case, would make no allowance for the effect of rotation. It is evident that the flexural rigidity, in a disc of which the centre is clamped, must in practice prevent the occurrence of modes in which the displacement, or its slope, is discontinuous at the centre, and will in this way bring the centrifugal system into operation. Special calculations are therefore required for the symmetrical modes, and the gravest frequency has been investigated (Section II b, § 5) by means of Rayleigh's method of an assumed type,* in which the potential and kinetic energies are calculated on the basis of an assumed form of w, and the frequencies determined from the condition that their sum must be constant in the vibration. It is known that this method gives an upper limit to the true value, and hence it may also be applied to modes having nodal diameters, as a convenient means of estimating the limits of possible error involved in the use of the formula (34A).

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* In these, $\partial w/\partial r$ has finite values at the centre when the disc is completely free.

Although it still has no direct bearing on the occurrences in actual turbine discs, it is evident that our analysis has been made more representative of practical conditions by the assumption of central clamping. The new frequencies found for the symmetrical modes are of special interest, since it appears that vibrations of this type have been a source of trouble in practice.†

Exception may perhaps be taken to the fact that in this, as in the former paper, certain factors which would require consideration in a strict theory of flexure have been ignored: no account has been taken of the additional deflections which accompany the transverse shear stresses, or of the terms which depend on "rotatory inertia." Both of these factors increase in importance with the thickness of the disc, and occur whether the disc rotates or not. In the latter case they are generally neglected, and are in fact quite unimportant except in the higher modes of vibration, where the disc is divided by nodal lines into a large number of segments. When the disc rotates, the terms which depend upon "rotatory inertia" are modified by the addition of terms of "gyroscopic" type, containing the angular velocity ω as a factor. It is not difficult to show that these cancel out of the general differential equation of transverse vibration, but they appear in the boundary conditions. On the other hand, in the problems of this and the former paper they prove to be of the same order of magnitude as the other "rotatory inertia" terms (of ordinary type), and may therefore be equally neglected.

The author desires to acknowledge his indebtedness to Miss B. S. Gough and Miss S. W. Skan, of the Aeronautics Department, National Physical Laboratory, for the valuable assistance they have given him in the detailed calculations of this paper; also to Miss C. G. C. Phillips, who prepared the diagrams for reproduction.

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* 'Theory of Sound,' vol. 1, §§ 88, 89.

† *Cf.* Baumann, *loc. cit.*

I. Free Transverse Vibrations of a Non-Rotating Circular Disc Clamped at its Centre.

§ 2. Following Kirchhoff and Rayleigh, we shall neglect the effects of "rotatory inertia" and of the additional deflection caused by shear. Then the differential equation which governs the transverse displacement (w) of the middle-surface may be written, in polar co-ordinates, as follows*:—

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\mathrm{E}h^2}{3(1-\sigma^2)} \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]^2 w = 0. \tag{1}$$

In this equation, ρ denotes the density of the material, E Young's Modulus, and σ Poisson's Ratio, while h is the half-thickness of the plate.

The boundary conditions of the present problem are

$$\left.\begin{aligned} &\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = 0, \\ &\frac{\partial}{\partial r}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] + \frac{(1-\sigma)}{r^2}\frac{\partial^2}{\partial \theta^2}\left[\frac{\partial w}{\partial r} - \frac{w}{r}\right] = 0, \end{aligned}\right\} \tag{2}$$

at the free edge ($r = a$), and

$$\left.\begin{aligned} w &= 0, \\ \frac{\partial w}{\partial r} &= 0, \end{aligned}\right\} \text{ at the circumference of the clamping circle } (r = b). \tag{3}$$

Assuming a solution of the type

$$w = \mathrm{W} \sin s(\theta - \theta_0) \sin p(t - t_0), \tag{4}$$

and substituting in (1), we obtain the equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{s^2}{r^2}\right]^2 \mathrm{W} - \kappa^4 \mathrm{W} = 0, \tag{5}$$

where

$$\kappa^4 = \frac{3(1-\sigma^2)\rho p^2}{\mathrm{E}h^2}. \tag{6}$$

The complete expression for W thus consists of the two solutions of the equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{s^2}{r^2} + \kappa^2\right] \mathrm{W} = 0, \tag{7}$$

together with the two solutions of the equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{s^2}{r^2} - \kappa^2\right] \mathrm{W} = 0. \tag{8}$$

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* *Cf.* Rayleigh, 'Theory of Sound,' vol. 1, § 218, or Love, 'Theory of Elasticity,' § 314 (d).

The solutions of (7) are the Bessel functions denoted by $J_s(\kappa r)$ and $Y_s(\kappa r)$, and the solutions of (8) are the modified Bessel functions denoted by $I_s(\kappa r)$ and $K_s(\kappa r)$. Thus we have

$$W = \alpha J_s(\kappa r)+\beta Y_s(\kappa r)+\gamma I_s(\kappa r)+\delta K_s(\kappa r), \tag{9}$$

where α, β, γ and δ are arbitrary coefficients.

Then if we employ the recurrence formulæ for Bessel functions, viz.,

$$\left.\begin{aligned} J_n'(z) &= \frac{n}{z}J_n(z)-J_{n+1}(z), \\ I_n'(z) &= \frac{n}{z}I_n(z)+I_{n+1}(z), \end{aligned}\right\} \tag{10}$$

and the similar formulæ for $Y_n'(z)$ and $K_n'(z)$ respectively,* the edge conditions (2) may be written in the form†

$$\begin{aligned} \kappa^2\Bigg(\alpha&\left[J_s(\kappa a)-(1-\sigma)\left\{\frac{s(s-1)}{\kappa^2a^2}J_s(\kappa a)+\frac{1}{\kappa a}J_{s+1}(\kappa a)\right\}\right] \\ +\beta&\left[Y_s(\kappa a)-(1-\sigma)\left\{\frac{s(s-1)}{\kappa^2a^2}Y_s(\kappa a)+\frac{1}{\kappa a}Y_{s+1}(\kappa a)\right\}\right] \\ -\gamma&\left[I_s(\kappa a)+(1-\sigma)\left\{\frac{s(s-1)}{\kappa^2a^2}I_s(\kappa a)-\frac{1}{\kappa a}I_{s+1}(\kappa a)\right\}\right] \\ -\delta&\left[K_s(\kappa a)+(1-\sigma)\left\{\frac{s(s-1)}{\kappa^2a^2}K_s(\kappa a)-\frac{1}{\kappa a}K_{s+1}(\kappa a)\right\}\right]\Bigg)=0 \end{aligned} \tag{11}$$

and

$$\begin{aligned} \kappa^2\Bigg(\alpha&\left[sJ_s(\kappa a)-\kappa a J_{s+1}(\kappa a)+\frac{(1-\sigma)s^2}{\kappa^2a^2}\left\{(s-1)J_s(\kappa a)-\kappa a J_{s+1}(\kappa a)\right\}\right] \\ +\beta&\left[sY_s(\kappa a)-\kappa a Y_{s+1}(\kappa a)+\frac{(1-\sigma)s^2}{\kappa^2a^2}\left\{(s-1)Y_s(\kappa a)-\kappa a Y_{s+1}(\kappa a)\right\}\right] \\ -\gamma&\left[sI_s(\kappa a)+\kappa a I_{s+1}(\kappa a)-\frac{(1-\sigma)s^2}{\kappa^2a^2}\left\{(s-1)I_s(\kappa a)+\kappa a I_{s+1}(\kappa a)\right\}\right] \\ -\delta&\left[sK_s(\kappa a)+\kappa a K_{s+1}(\kappa a)-\frac{(1-\sigma)s^2}{\kappa^2a^2}\left\{(s-1)K_s(\kappa a)\right.\right. \\ &\qquad\qquad\left.\left.+\kappa a K_{s+1}(\kappa a)\right\}\right]\Bigg)=0, \end{aligned} \tag{12}$$

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* Whittaker and Watson, 'Modern Analysis,' §§ 17.21, 17.61, 17.7, and 17.71.

† The coefficients κ and κ^2 are important, since the equations may be satisfied solely by reason of κ being zero : a being necessarily finite, equation (12) has been multiplied throughout by this quantity.

whilst the conditions (3) at the circumference of the clamping circle become

$$\alpha \mathrm{J}_s(\kappa b)+\beta \mathrm{Y}_s(\kappa b)+\gamma \mathrm{I}_s(\kappa b)+\delta \mathrm{K}_s(\kappa b) = 0 \tag{13}$$

and

$$\kappa\left\{\alpha\left[\frac{s}{\kappa b}\mathrm{J}_s(\kappa b)-\mathrm{J}_{s+1}(\kappa b)\right]+\beta\left[\frac{s}{\kappa b}\mathrm{Y}_s(\kappa b)-\mathrm{Y}_{s+1}(\kappa b)\right]\right.$$
$$\left.+\gamma\left[\frac{s}{\kappa b}\mathrm{I}_s(\kappa b)+\mathrm{I}_{s+1}(\kappa b)\right]+\delta\left[\frac{s}{\kappa b}\mathrm{K}_s(\kappa b)+\mathrm{K}_{s+1}(\kappa b)\right]\right\} = 0. \tag{14}$$

Eliminating the coefficients α, β, γ, δ from (11)—(14), we obtain a determinantal equation which, for any required value of the ratio b/a, may be solved by trial,* assumed values of κa being adopted, and the resulting values of the determinant plotted against these. Table I gives results obtained in this way in respect of the gravest frequencies for modes in which s has the values 0, 1, 2 and 3: σ has been taken to be 0·3.

Table I.

$s = 0$.		$s = 1$.		$s = 2$.		$s = 3$.	
b/a.	κa.	b/a.	κa.	b/a.	κa.	b/a.	κa.
0·276	2·50	0·060	1·68	0·186	2·50	0·43	4·0
0·642	5·00	0·397	3·00	0·349	3·00	0·59	5·0
0·840	9·00	0·603	4·60	0·522	4·00	0·71	7·0
		0·634	5·00	0·769	8·00	0·82	10·0
		0·771	8·00	0·81	10·00		
		0·827	11·00				

In the limiting case, when the radius of the clamping circle is indefinitely reduced, the analysis may be considerably simplified. For if s has a value greater than 1, the coefficients of α and β in (13) and (14) vanish when $b = 0$, and these relations will accordingly be satisfied by making γ and δ zero, without restriction of α or β. The frequencies are thus found by eliminating α and β between (11) and (12), which now reduce to the corresponding equations of Kirchhoff for a completely free disc.

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* Tables of the functions $\mathrm{J}_n(z)$ and $\mathrm{I}_n(z)$ (n integral) are given by Gray and Matthews, 'Treatise on Bessel Functions,' pp. 247–288. The Neumann functions $\mathrm{Y}_0(z)$ and $\mathrm{Y}_1(z)$, as here defined, have been tabulated by the British Association, 'Report for 1914,' pp. 75–82, and the functions $\mathrm{K}_0(z)$ and $\mathrm{K}_1(z)$ by W. S. Aldis, 'Roy. Soc. Proc.,' A, vol. 64, pp. 203–223 (1899).

When s has the values 1 or 0, we may substitute in (13) and (14) from the ascending series for the Bessel functions. We have

$$\left.\begin{array}{l}
J_0(z) = 1-\frac{z^2}{4}+\ldots, \qquad J_1(z) = \frac{z}{2}-\frac{z^3}{16}+\ldots, \\
I_0(z) = 1+\frac{z^2}{4}+\ldots, \qquad I_1(z) = \frac{z}{2}+\frac{z^3}{16}+\ldots, \\
Y_0(z) = J_0(z)\log z+\frac{z^2}{4}+\ldots, \\
Y_1(z) = J_1(z)\log z-\frac{1}{z}-\frac{z}{4}+z^3\ldots, \\
K_0(z) = -I_0(z)\{\log z-E\}+\frac{z^2}{4}+\ldots, \\
K_1(z) = -I_1(z)\{\log z-E\}-\frac{1}{z}+\frac{z}{4}+\ldots. \\
\text{where* } \quad E = \log 2 - \gamma\,(\gamma \text{ being Euler's constant}), \\
\qquad = 0{\cdot}11593\ldots
\end{array}\right\} \qquad (15)$$

Hence (observing that the limit of $z\log z$, when $z \to 0$, is zero) we see that the significant terms in (13) and (14), when $s = 1$, are

$$\left.\begin{array}{l}
-\frac{1}{\kappa b}(\beta+\delta) = 0, \\
\text{and} \quad \frac{\alpha+\gamma}{2}+\beta\left(\frac{\log\kappa b}{2}+\frac{1}{\kappa^2 b^2}+\frac{1}{4}\right)-\delta\left(\frac{\log\kappa b-E}{2}-\frac{1}{\kappa^2 b^2}+\frac{1}{4}\right) = 0,
\end{array}\right\} \qquad (16)$$

respectively. It follows that $(\beta+\delta)$ must vanish, and that both β and δ must be infinitesimal in comparison with α and γ, although their contribution to the value of the *slope* is finite at the centre. Hence, if κa is finite, the contributions of β and δ to (11) and (12) will be infinitesimal, and the finite frequencies of vibration in modes characterized by one nodal diameter can be found from a relation identical with that which obtains for a completely free disc; that is to say, these frequencies also are unaffected by clamping which is entirely confined to the centre.

The gravest frequency for the completely free disc, in modes characterized by one nodal diameter, is zero, the "vibration" consisting simply in a rigid-body rotation of the disc about a diameter. We should expect that this zero frequency will become finite when the slope of the disc is constrained at the centre; but it is easily verified that equations (11)—(14), when $s = 1$, are satisfied if we write

$$\kappa a = \kappa b = 0,$$

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[*] This significance for E (the symbol employed in the tables referred to) continues down to equation (24), after which E is again used to denote Young's Modulus.

so that a zero frequency of vibration is still possible in the limiting case, when the radius of the clamping circle is indefinitely reduced. Alternatively, the existence of a mode having zero frequency can be established by verifying that the expression

$$w = (\epsilon r + \zeta r \log r) \sin(\theta - \theta_0), \tag{17}$$

where ϵ and ζ are arbitrary constants, is a solution of equation (1), which will also satisfy (2) and (3), provided that ζ is infinitesimal, but related to ϵ by the condition

$$\epsilon + \zeta[\log r + 1]_{r=0} = 0. \tag{18}$$

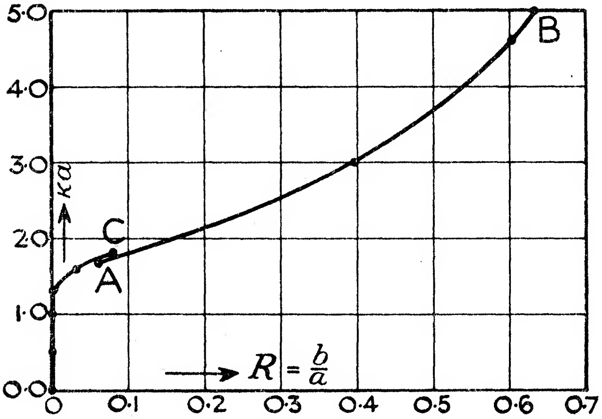

Fig. 1.

On the other hand, when the results (Table I) obtained for finite values of b/a are plotted, as in the curve AB of fig. 1, κa appears to be tending to a finite value as $b/a \to 0$. It thus appears that the frequency must rise very rapidly as the radius of the clamping circle increases from a zero value, and the conclusion may be confirmed by expanding the determinantal equation which is obtained by eliminating α, β, γ, δ from (11)—(14), on the assumption that κb is very small. When $s = 1$, the determinantal equation is easily thrown into the form*

$$A^4 \begin{vmatrix} AJ_1(A) - \eta J_2(A), & AY_1(A) - \eta Y_2(A), & AI_1(A) - \eta I_2(A), & AK_1(A) - \eta K_2(A), \\ J_2(A), & Y_2(A), & -I_2(A), & -K_2(A), \\ BJ_1(B), & BY_1(B), & -BI_1(B), & -BK_1(B), \\ J_0(B), & Y_0(B), & -I_0(B), & -K_0(B), \end{vmatrix} = 0, \tag{19}$$

---

* a being finite, it is legitimate to multiply the first two rows by this quantity. The complete equation contains an additional factor $1/b$, which has been cancelled out of equation (19), since it cannot be zero.

in which A and B are written for κa and κb respectively, and η is substituted (for brevity) in place of the quantity $(1-\sigma)$. If we expand the determinant, substituting from (15) for the Bessel functions of argument B, the positive powers of B are found to be associated with powers of A not lower than the fourth, whilst the terms which are important when $B \to 0$ may be obtained from the determinantal equation

$$A^4 \begin{vmatrix} AJ_1(A)-\eta J_2(A), & AY_1(A)-\eta Y_2(A), & AI_1(A)-\eta I_2(A), & AK_1(A)-\eta K_2(A), \\ J_2(A), & Y_2(A), & -I_2(A), & -K_2(A), \\ 0, & -1, & 0, & 1, \\ 1, & \log B, & -1, & (\log B-E), \end{vmatrix} = 0. \qquad (20)$$

Hence we find that the complete equation (19) may be written in the form

(positive powers of B) × (terms in A^4 and higher powers of A)

$$-[(3+\sigma)A^4+(\text{higher powers of A})]$$
$$-(2\log B-E)\left[\frac{A^8}{32}(3+\sigma)+(\text{higher powers of A})\right]=0,$$

and when $A \to 0$ the significant terms in this equation are

$$-(3+\sigma)A^4\left(1+\frac{A^4}{16}\log B\right)=0. \qquad (21)$$

The solution $A = 0$ (for all values of B) corresponds to the configuration of equilibrium, with which we are not at present concerned; the alternative solution may be written (since $A\log A \to 0$ with A) in the form

$$A^4 = -\frac{16}{\log R}, \text{ where } R = b/a. \qquad (22)$$

Hence, $A \to 0$ when $R \to 0$ (the result found previously), but

$$\frac{A}{R} = 2\sqrt[4]{\left(\frac{-1}{R^4\log R}\right)},$$

so that $\frac{A}{R}\left(\text{or } \frac{dA}{dR}\right) \to \infty$ as $R \to 0$.

The diagram shown in fig. 1 connects values of A (or κa) with values of R (or b/a); AB is a curve drawn through points which have been plotted from Table I, and OC is a curve obtained from the approximate determinantal equation (20), tables being employed to find the exact values of the Bessel functions in the first and second rows. In this connection, it is convenient to employ the substitutions

$$AJ_2(A) = 2J_1(A)-AJ_0(A),$$
$$AY_2(A) = 2Y_1(A)-AY_0(A),$$
$$-AI_2(A) = 2I_1(A)-AI_0(A),$$
$$-AK_2(A) = 2K_1(A)-AK_0(A).$$

From the fact that the two curves very nearly overlap, it is evident that (20) approximates closely to the correct equation over the range considered; a sufficiently exact transition from one curve to the other may be made by eye.

This completes the discussion of clamping for modes in which $s = 1$. When $s = 0$, the significant terms in (13) and (14) are

$$\left.\begin{aligned} \alpha+\gamma+(\beta-\delta)\log\kappa b+\delta\mathrm{E} &= 0 \\ \text{and} \qquad \frac{\beta-\delta}{\kappa b} &= 0. \end{aligned}\right\} \tag{23}$$

Both relations are satisfied if we write

$$\beta = \delta = -\frac{1}{\mathrm{E}}(\alpha+\gamma), \tag{24}$$

and (11) and (12) can then be expressed in terms of α and γ only. Eliminating α and γ, we obtain an equation in κa which may be solved, as before, by trial. The results depend to some extent upon the value of σ, which throughout this paper has been taken to be 0·3 (a representative value for steel); for this value, the first four calculated roots for κa are

$$\kappa a = 1{\cdot}937,\ 4{\cdot}573,\ 7{\cdot}79,\ 10{\cdot}94,$$

and the corresponding frequencies can be obtained from (6) in the form

$$p^2 = \frac{\mathrm{E}h^2}{\rho a^4} \times (5{\cdot}156,\ 160{\cdot}2,\ 1349,\ 5250), \tag{25}$$

where E is Young's Modulus for the material considered.

Fig. 2 illustrates the nature of the two gravest modes.

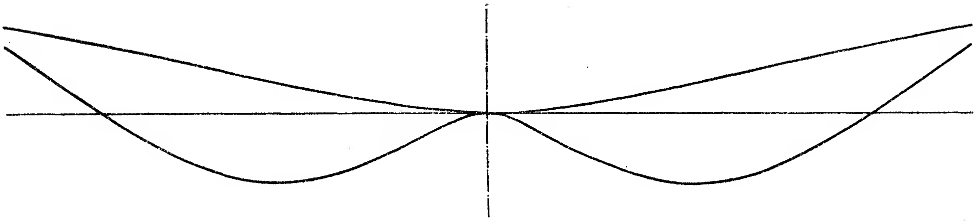

Fig. 2.

The variation of the frequency with the radius of the clamping circle, as found from the foregoing analysis, is exhibited in figs. 3–6, by means of curves which connect $p\sqrt{\left(\frac{\rho a^4}{\mathrm{E}h^2}\right)}$ (a non-dimensional quantity proportional to the frequency) with the ratio $R\,(= b/a)$; these show, as we should expect from physical considerations, that the effects of clamping are unimportant for small values of R, unless $s = 1$, when they increase very rapidly with the radius of the clamping circle.

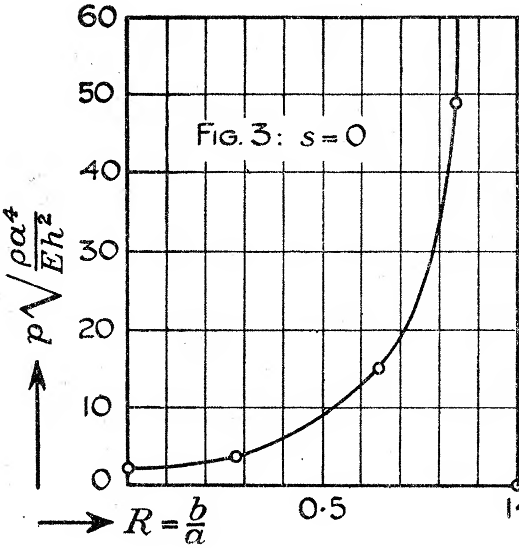
Fig. 3: $s=0$

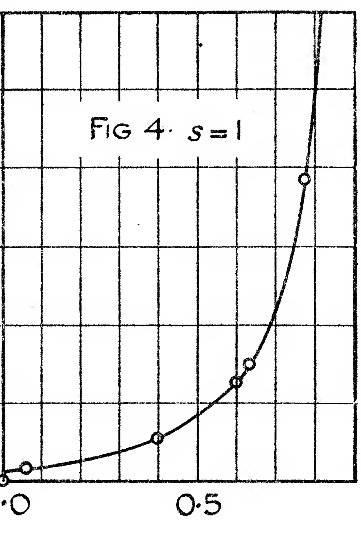
Fig 4· $s=1$

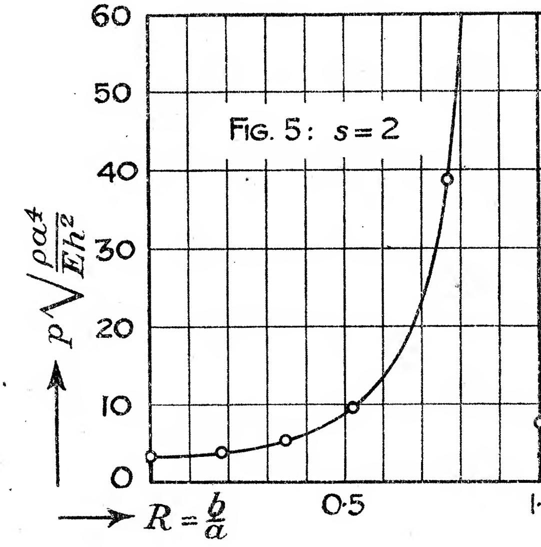
Fig. 5: $s=2$

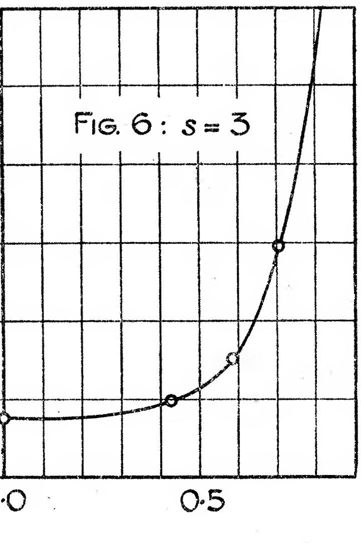
Fig. 6: $s=3$

II. THE EFFECTS OF ROTATION.

(a) *Flexural Rigidity Neglected.*

§ 3. As in the paper on "Spinning Discs,"* we begin by considering the extreme case, in which the disc is so thin, and the rotation so rapid, that the flexural forces are negligible. Under these conditions, the equation of transverse motion may be written in the form*

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* *Loc. cit.*, § 2. On the assumption stated in the introductory section of this paper, the centrifugal stress-system, and hence the equation of transverse motion, will be unaffected by clamping.

$$A(a^2-r^2)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)-2Ar\frac{\partial w}{\partial r}+\left(\frac{p^2}{\omega^2}+s^2B-\frac{s^2Aa^2}{r^2}\right)w=0, \qquad (5\text{A})$$

where $$A=\frac{1}{8}(3+\sigma), \qquad B=\frac{1}{8}(1+3\sigma). \qquad (2\text{A})$$

If we write

$$\frac{p^2}{\omega^2}+s^2B=m(m+2)A, \qquad (8\text{A})$$

and $$x=\frac{r^2}{a^2},$$

equation (5A) may be reduced to the form

$$4\frac{\partial}{\partial x}\left\{x(1-x)\frac{\partial w}{\partial x}\right\}+\left\{m(m+2)-\frac{s^2}{x}\right\}w=0. \qquad (26)$$

Again, if we write

$$w=yx^{s/2}$$

in (26), we obtain, as the differential equation to be satisfied by y,

$$4x(1-x)\frac{d^2y}{dx^2}+4\{1+s-(s+2)x\}\frac{dy}{dx}+\{m(m+2)-s(s+2)\}y=0,$$

and this may be written in the form

$$\frac{d^2y}{dx^2}+\frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\frac{dy}{dx}-\frac{\alpha\beta}{x(1-x)}y=0, \qquad (27)$$

where $$\alpha=\tfrac{1}{2}(s-m), \qquad \beta=\tfrac{1}{2}(s+m+2), \qquad \gamma=s+1. \qquad (12\text{A})$$

Equation (27) is hypergeometric. The solution finite for $x=0$ is

$$\left.\begin{array}{l} y_1=C\left\{1+\dfrac{\alpha\beta}{1.\gamma}x+\dfrac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\ldots\right\}, \\ \qquad\qquad \text{where C is an arbitrary constant,} \\ \text{or} \qquad y_1=CF(\alpha,\beta,\gamma,x) \end{array}\right\} \qquad (28)$$

(in Gauss' notation). This is the solution which was discussed in the paper on "Spinning Discs," and the natural frequencies were obtained from the known theorem that, since

$$\gamma=\alpha+\beta, \qquad (13\text{A})$$

the expansion of $F(\alpha, \beta, \gamma, x)$ becomes logarithmically infinite at the edge ($x=1$), unless it terminates.† Hence we must have

$$m=s+2n, \qquad (14\text{A})$$

where n is a positive integer.

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* The suffix previously attached to p is for the present omitted.

† Forsyth, 'Differential Equations,' § 113.

To take account of constraints which prevent transverse displacement along some circle (x = constant), we must consider the second solution of (27). When s is finite, this may be written in the form*

$$\left.\begin{aligned} y_2 &= \mathrm{C}' x^{1-\gamma} \mathrm{F}(\alpha', \beta', \gamma', x), \\ \text{where} \quad \alpha' &= \alpha+1-\gamma = -\tfrac{1}{2}(s+m), \\ \beta' &= \beta+1-\gamma = \tfrac{1}{2}(m+2-s), \\ \gamma' &= 2-\gamma = 1-s, \end{aligned}\right\} \tag{29}$$

whilst if $s = 0$ (so that $\gamma = 1$) the second solution is†

$$y_2 = \mathrm{C}'\left[\mathrm{F}(\alpha, \beta, 1, x) \log x + \left\{\frac{\alpha\beta}{1^2}\left(\frac{1}{\alpha}+\frac{1}{\beta}-\frac{2}{1}\right)x \right.\right.$$
$$\left.\left. + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1^2 \, . \, 2^2}\left(\frac{1}{\alpha}+\frac{1}{\alpha+1}+\frac{1}{\beta}+\frac{1}{\beta+1}-\frac{2}{1}-\frac{2}{2}\right)x^2+\ldots\right\}\right], \tag{30}$$

C′ denoting in each instance an arbitrary constant. Now, since

$$\gamma' = \alpha'+\beta',$$

the expansion of $\mathrm{F}(\alpha', \beta', \gamma', x)$, in (29), is divergent unless it terminates. But it is easily seen, from (28), that if $\mathrm{F}(\alpha, \beta, \gamma, x)$ terminates with the term in x^n, so that (14) A is satisfied, then $\mathrm{F}(\alpha', \beta', \gamma', x)$ will terminate with the term in x^{s+n}, so that the solution given by (29) consists of a finite number of terms, beginning with a term in x^{-s} and ending with a term in x^n.

Again, it is evident that the quantity in twisted brackets on the right of (30) will terminate under the same conditions as $\mathrm{F}(\alpha, \beta, \gamma, x)$, and at a term of the same order in x; moreover, the factor $\mathrm{F}(\alpha, \beta, 1, x)$ in the first term is the solution found in (28). Hence, for all values of s, by suitably choosing the ratio of C′ to C in the complete solution

$$y = y_1+y_2,$$

we can make y (and hence w) vanish at any given radius; and *although the mode will be altered, the frequency of vibration will be unchanged.* When the constraint is applied at the centre, so that the radius at which w vanishes is indefinitely reduced, it is clear that the ratio of C′ to C will be infinitesimal, so that the mode is unaffected by the constraint, except at the centre: at the centre it will be affected only if $s = 0$, since for other values of s the central displacement is zero in the first solution (28).

(b) *Flexural and Centrifugal Stresses both Operative.*

§ 4. When the centrifugal tensions and the flexural stiffness are both operative, the difficulties of an exact solution are much increased, since the modes as well as the frequencies of vibration depend upon the relative importance of the two systems. In regard to the gravest frequency corresponding to any given value of s, it was shown in the paper on "Spinning Discs"* that a *lower limit* to the frequency p can be obtained from the equation

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* Forsyth, *op. cit.*, § 115.

† Forsyth, *op. cit.*, p. 251, Ex. 1.

$$p^2 = p_1^2 + p_2^2, \tag{34A}$$

where p_1 and p_2 are the values of p obtained by neglecting the flexural rigidity, and the effects of rotation, respectively.

The discussion just given shows that the values of p_1^2 are unaffected by clamping. Thus we may write, as in the earlier paper,

$$p_1^2 = \lambda\omega^2, \tag{20A}$$

where λ has values which may be found from the relations (2A), (8A) and (14A), given above. The following table (taken from the earlier paper, and amplified by the inclusion of a column for modes in which $s = 0$) gives numerical values for λ on the hypothesis that $\sigma = 0{\cdot}3$, the value assumed previously:—

Table II.—Values of λ in the formula $p_1^2 = \lambda\omega^2$.

(N.B.—These values are not affected by clamping.)

	$s = 0$.	$s = 1$.	$s = 2$.	$s = 3$.
$n = 0$	0	1	2·35	4·05
$n = 1$	3·3	5·95	8·95	12·3
$n = 2$	9·9	14·2	18·85	23·85
$n = 3$	19·8	25·75	32·05	38·7

The results of Section I show that the radius of clamping has an influence upon the value of p_2^2, but that when this radius is indefinitely reduced the only frequencies affected by clamping are those corresponding to the modes in which $s = 0$. Hence, writing

$$p_2^2 = \mu \frac{Eh^2}{\rho a^4}, \tag{21A}$$

as in the "Spinning Discs" paper, we have the following table of values for μ, of which only the second column ($s = 0$, centre clamped) is new: σ is again assumed to have the value 0·3.

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* *Loc. cit.*, § 4.

Table III.—Values of μ in the formula $p_2^2 = \mu \dfrac{Eh^2}{\rho a^4}$.

(N.B.—Clamping is confined to the centre of the Disc.)

	$s = 0$.		$s = 1$.	$s = 2$.	$s = 3$.
	Centre free.	Centre clamped.	(These values are not affected by central clamping.)		
$n = 0$	0	5·16	0*	10·61	56·87
$n = 1$	29·67	160·2	154·3	443·9	1040
$n = 2$	577	1349	1222	2624	4285
$n = 3$	2983	5250	5266	8574	13287

* But *cf.* the discussion of this case given in §2. When the radius of clamping is finite, though very small, 1·05 may be taken as a representative value of μ.

By means of these tables (or of Table II and figs. 3—6, when the radius of clamping is finite) the lower limit given by (34A) can be quickly calculated. But it is evident that no allowance will be made in this way for the effect of rotation in regard to the modes in which $s = 0$: these, in a disc of which the centre is clamped, must be conditioned in the main by the flexural system, and will necessarily be such as will bring the centrifugal system into operation.

§ 5. In the "Spinning Discs" paper,* closely approximate results were obtained by means of Rayleigh's method of an assumed type,† in which the potential and kinetic energies are calculated on the basis of an assumed expression for w, and the frequencies determined from the condition that their sum remains constant in the vibration. A frequency thus calculated is stationary for slight variations of the type, and a good approximation can, therefore, be obtained in this way if the type be suitably chosen. Moreover, in the case of the gravest mode of all, it is known that the frequency will be overestimated, and in the present problem this statement applies also to the gravest mode corresponding to any definite value of s. Prof. Lamb has made use of this fact in devising a method whereby the accuracy of Rayleigh's method may be improved; it consists in the inclusion of a variable parameter in the assumed expression for w, the value of this parameter being afterwards adjusted so as to make the calculated frequency a minimum.‡ Using this device, the gravest frequency of vibration in a mode having two nodal diameters was estimated, for a non-rotating disc, with an error less than $\frac{1}{2}$ per cent.*

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* *Loc. cit.*, p. 276.

† 'Theory of Sound,' vol. 1, §§ 88, 89.

‡ *Cf.* his paper "On the Vibrations of an Elastic Plate in Contact with Water," 'Roy. Soc. Proc.,' A, vol. 98, pp. 205–216 (1920), which appears to be the earliest illustration of the method.

Applying these methods to the determination of the gravest frequency in modes for which $s = 0$, we have as expressions for the potential and kinetic energies, in polar co-ordinates† :—

V_1 = Potential Energy of the Centrifugal Forces,

$$= 2\pi h \int_0^a P \left(\frac{\partial w}{\partial r}\right)^2 r dr, \tag{31}$$

where $$P = A(a^2 - r^2)\rho\omega^2, \tag{1A}$$

and A is given by equation (2A) above ;

V_2 = Potential Energy of Flexure,

$$= \frac{2\pi E h^3}{3(1-\sigma^2)} \int_0^a \left\{ \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right]^2 - \frac{2(1-\sigma)}{r} \frac{\partial w}{\partial r} \cdot \frac{\partial^2 w}{\partial r^2} \right\} r dr \, ; \tag{32}$$

T = Kinetic Energy of the Motion,

$$= 2\pi\rho h \int_0^a \left(\frac{\partial w}{\partial t}\right)^2 r dr. \tag{33}$$

In the application of Rayleigh's method, an assumed form for w is substituted in these expressions, and we obtain an estimate of the frequency from the energy equation

$$V_1 + V_2 + T = \text{constant}. \tag{34}$$

We begin with the extreme case in which ω is zero, since accurate results are here available for comparison. The simplest expression for w which satisfies the conditions that w and $\partial w / \partial r$ vanish at the centre is

$$w = k \frac{r^2}{a^2} \sin p_2 t, \tag{35}$$

where k is an arbitrary constant. Substituting this expression in (32) and (33), we have

$$\left.\begin{aligned} V_2 &= \frac{8\pi k^2 E h^3}{3(1-\sigma)a^2} \sin^2 p_2 t, \\ T &= \frac{\pi}{3} k^2 p_2^2 \rho h a^2 \cos^2 p_2 t, \end{aligned}\right\}$$

and from (34) we obtain the value

$$p_2^2 = \frac{8}{1-\sigma} \cdot \frac{E h^2}{\rho a^4},$$

whence $$\mu = 11{\cdot}429, \text{ when } \sigma = 0{\cdot}3. \tag{36}$$

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* *Vide* § 5 of the "Spinning Discs" paper, equation (52).

† *Cf.* equations (25)–(29) of the "Spinning Discs" paper.

Comparing this result with the correct figure (5·16) given in Table III, we see that it is over 100 per cent. in error. To obtain a closer approximation, we may employ the artifice of Prof. Lamb, and assume, in place of (35), that

$$w = \alpha\frac{r^2}{a^2}\left(1+\beta\frac{r^2}{a^2}\right)\sin p_2 t, \tag{37}$$

α and β being arbitrary constants. Then we have

$$\left.\begin{aligned} V_2 &= \frac{8\pi\alpha^2 Eh^3}{3(1-\sigma^2)a^2}\left\{(1+\sigma)(1+4\beta)+\frac{4}{3}(5+3\sigma)\beta^2\right\}\sin^2 p_2 t, \\ T &= \pi\alpha^2 p_2^2\,\rho h a^2\left\{\frac{1}{3}+\frac{\beta}{2}+\frac{\beta^2}{5}\right\}\cos^2 p_2 t, \end{aligned}\right\}$$

and (34) gives the expression

$$p_2^2 = \frac{8}{3(1-\sigma^2)}\,\frac{Eh^2}{\rho a^4}\left\{\frac{(1+\sigma)(1+4\beta)+4/3\,(5+3\sigma)\,\beta^2}{1/3+\beta/2+\beta^2/5}\right\}. \tag{38}$$

Thus we have, in equation (21A) above, the expression

$$\mu = \frac{L+2M\beta+N\beta^2}{l+2m\beta+n\beta^2}, \tag{39}$$

where, if $\sigma = 0·3$,

$$\left.\begin{aligned} L &= \frac{8}{3(1-\sigma)} = 3·81, & l &= \frac{1}{3}, \\ M &= \frac{16}{3(1-\sigma)} = 7·62, & m &= \frac{1}{4}, \\ N &= \frac{32(5+3\sigma)}{9(1-\sigma^2)} = 23·06, & n &= \frac{1}{5}. \end{aligned}\right\} \tag{40}$$

The stationary values of μ are given by the quadratic

$$(ln-m^2)\,\mu^2-(lN+nL-2mM)\,\mu+(LN-M^2) = 0,$$

or

$$\frac{1}{240}\,\mu^2-4·64\,\mu+29·79 = 0,$$

whence the gravest value of p_2^2, in (38), is given by

$$p_2^2 = 6·457\,\frac{Eh^2}{\rho a^4}, \text{ approximately,} \tag{41}$$

a result which is still 25 per cent. in error.

It is evident that the additional boundary conditions at the centre of the disc increase the difficulty of finding a suitable assumed form for w. The value of β which corresponds to (41) is $-0·275$ approximately, and when this is substituted in (37) the resulting form for w is found to differ materially from the correct form shown in fig. 2. After several failures to guess a better form, the plan was adopted of calculating the deflection of the disc when clamped at its centre and subjected to a loading varying in intensity as some power of the radial distance, the edge being assumed to be free from shearing forces, but not necessarily from stress-couples. This deflection, multiplied by $\sin p_2 t$, was taken as the assumed form of w, and the frequency was then calculated by Rayleigh's method: finally, the index of r in the assumed intensity of loading was chosen so as to make the calculated frequency a minimum. In this way, the advantages of Prof. Lamb's device are retained, whilst the assumed mode, independently of the value given to the variable parameter, satisfies three out of the four boundary conditions.

It is easily verified that the expression

$$W = C\left\{r^q - \frac{q^2(q-2)}{4}a^{q-2}r^2\log\frac{r}{a}\right\}, \tag{42}$$

where C is an arbitrary constant, gives the displacement corresponding to a loading of intensity

$$Z = D\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right]^2 W = CDq^2(q-2)^2r^{q-4}, \tag{43}$$

where $D = \frac{2}{3}\frac{Eh^3}{1-\sigma^2}$; it may also be shown to satisfy the second of the edge conditions (2), together with both of the clamping conditions (3), provided that $q > 1$ and $q \neq 2$.

For a value of q close to 3 it will also satisfy the first of (2); but it is preferable to leave q indeterminate for the present, and to choose it subsequently so as to make the calculated frequency a minimum. We find that

$$\int_0^a r\left[\frac{d^2W}{dr^2}+\frac{1}{r}\frac{dW}{dr}\right]^2 dr = C^2q^4a^{2(q-1)}\left[\frac{1}{2(q-1)}-\frac{2(q-1)(q-2)}{q^2}+\frac{(q-2)^2}{4}\right],$$

and

$$\int_0^a \frac{dW}{dr}\cdot\frac{d^2W}{dr^2}dr = C^2q^2a^{2(q-1)}\left[\frac{1}{2}-\frac{q(q-2)}{4}+\frac{q^2(q-2)^2}{32}\right],$$

whence the potential energy of flexure in the mode

$$w = W\sin pt \tag{44}$$

is

$$V_2 = \frac{2\pi Eh^3}{3(1-\sigma^2)}C^2q^2a^{2(q-1)}\sin^2 pt\left[\frac{q^2}{2(q-1)}-2(q-1)(q-2)+\frac{q^2(q-2)^2}{4}\right.$$
$$\left.-(1-\sigma)\left\{1-\frac{q(q-2)}{2}+\frac{q^2(q-2)^2}{16}\right\}\right]. \tag{45}$$

The kinetic energy of the motion is

$$T = 2\pi\rho hp^2\cos^2 pt\int_0^a rW^2dr,$$
$$= 2\pi\rho hp^2C^2a^{2(q+1)}\cos^2 pt\left[\frac{1}{2(q+1)}+\frac{q^2(q-2)}{2(q+4)^2}+\frac{q^4(q-2)^2}{16.108}\right]. \tag{46}$$

The expression for the frequency is therefore given by equation (21A) above, if

$$\mu = \frac{q^2}{3(1-\sigma^2)} \times$$

$$\left[\frac{\dfrac{q^2}{2(q-1)} - 2(q-1)(q-2) + \dfrac{q^2(q-2)^2}{4} - (1-\sigma)\left\{1 - \dfrac{q(q-2)}{2} + \dfrac{q^2(q-2)^2}{16}\right\}}{\dfrac{1}{2(q+1)} + \dfrac{q^2(q-2)}{2(q+4)^2} + \dfrac{q^4(q-2)^2}{16\,.\,108}}\right] \qquad (47)$$

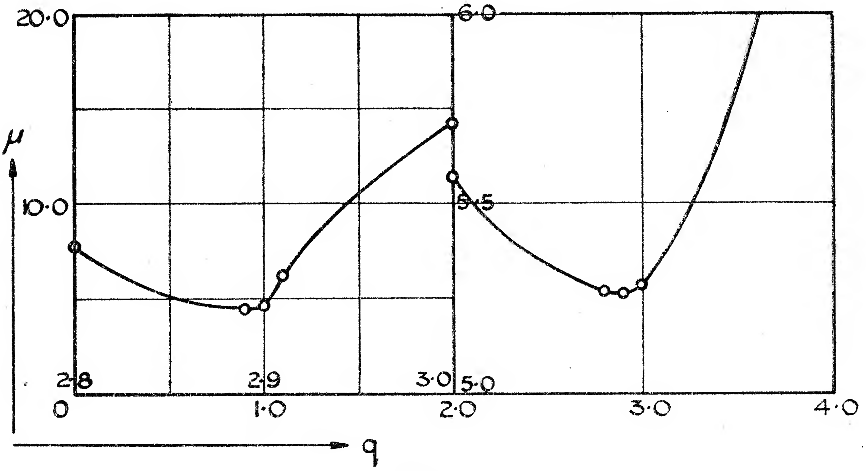

Fig. 7.

We have seen that the conditions to be satisfied at the centre require that q shall be greater than 1 and not equal to 2. Subject to this restriction, we may choose q so as to make μ, in (47), a minimum; the required value is best determined by plotting μ against q, as shown in fig. 7, where the important part of the curve is also plotted on a much enlarged scale. Taking σ as equal to 0·3, it is found that the expression μ has a minimum value of 5·22 when $q = 2·89$. This is an error of less than 1·2 per cent. in the value of μ, or less than 0·6 per cent. on the frequency, which is a very satisfactory result: but the rapid rise of μ, as q departs from the optimum value, serves to explain why such serious overestimates were obtained from what might have been considered reasonable assumptions for the form of w.

We proceed to consider the effects of rotation. Substituting for w, in (31), from (42) and (44), we have

$$V_1 = 2\pi\rho\omega^2 Ah\, C^2 q^2 a^{2q+2} \sin^2 pt \left[\frac{1}{2q} - \frac{q^2(q-2)}{2(q+2)^2} + \frac{q^2(q-2)^2}{8\,.\,16} - \left\{\frac{1}{2(q+1)} - \frac{q(q+2)(q-2)}{2(q+4)^2} + \frac{5q^2(q-2)^2}{3\,.\,18\,.\,16}\right\}\right].$$

The kinetic energy is given by (46), as before. Hence, by (34), the expression for the frequency will be of the form (20A), where

$$\lambda = \mathrm{A}q^2 \left[\frac{\dfrac{1}{2q(q+1)} - \dfrac{q(q-2)(q^2+2q-4)}{(q+2)^2(q+4)^2} + \dfrac{7q^2(q-2)^2}{16\,.\,27\,.\,8}}{\dfrac{1}{2(q+1)} + \dfrac{q^2(q-2)}{2(q+4)^2} + \dfrac{q^4(q-2)^2}{16\,.\,108}} \right] \tag{48}$$

λ, as determined from this expression, may be plotted against q, in the manner of fig. 7. Its minimum value is 0·416, and occurs when $q = 3{\cdot}295$: like that of μ, it is sharply defined, and the optimum value for q will clearly depend in the general case upon the relative importance of the flexural and centrifugal terms. But if we regard it as determined entirely by the flexural system, the value of λ will be only slightly, if at all, in excess of 0·5; adopting this value as sufficiently accurate for practical purposes, we may calculate the gravest frequency in a symmetrical mode from the formula

$$\left.\begin{aligned} & p^2 = \lambda\omega^2 + \mu\frac{\mathrm{E}h^2}{\rho a^4}, \\ \text{where} \quad & 0 < \lambda < 0{\cdot}5, \\ \text{and} \quad & 5{\cdot}16 < \mu < 5{\cdot}22. \end{aligned}\right\} \tag{49}$$

A closer approximation could, of course, be obtained, in any particular instance, by using the curves for λ and μ. Alternatively, by assuming the *mode* to be unchanged by rotation (so that μ is given by Table III), and employing (31), (33) and (34) for the calculation of λ, we could obtain upper limits to the value of this quantity, for use in the general formula (49): Table II, as we have seen, gives lower limits. The calculations would need to be done graphically.